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An asymptotic theory for the turbulent flow over a progressive water wave

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The turbulent flow over a progressive water wave is studied using an eddy viscosity model. The governing equations are treated asymptotically for the case $\epsilon \ll 1$, where ϵ is the square root of a characteristic drag coefficient. A calculation of the phase shift between the wave-induced pressure perturbation and the surface elevation shows that the phase shift is induced by a term in the gradient of the Reynolds stress. Growth rates are determined, and are shown to agree well with observations for the most rapidly amplifying waves. However, the present model and previous turbulence calculations are found to provide significantly lower growth rates than those measured by Snyder *et al.* (1981) for waves with phase velocities comparable to the wind speed.

1. Introduction

In the first of Miles's papers on wind-driven gravity waves (Miles 1957) the generation of deep-water waves by a vertically sheared wind in the x-direction is treated by expressing the air pressure at the water surface as the real part of

$$p = s\rho_{\mathrm{w}} V^2 \beta \delta \exp(\mathrm{i}\alpha(x-Ct)), \tag{1.1}$$

in which s is the ratio of the air density ρ to the water density ρ_{w}, α is the wavenumber, V is a characteristic velocity which we define as the wind speed at elevation $1/\alpha$, β is a complex pressure coefficient, δ is the wave slope, and C is a complex phase velocity. Using linear irrotational theory for treating the motion in the water yields

$$C^2 = c^2 + s\beta V^2, \tag{1.2}$$

where c is the phase velocity of irrotational deep-water waves, and the complex phase velocity is given approximately by

$$C = c\left[1 + \tfrac{1}{2}s\beta(V/c)^2\right]. \tag{1.3}$$

As noted by Miles, evaluating the pressure coefficient β at $C = c$ and substituting the result into (1.3) provides an approximation to the complex phase velocity correct up to and including $O(s)$ terms.

Miles's own work on the problem consists of a hybrid theory in which turbulence in the flow over the water is invoked to provide a logarithmic wind profile but is otherwise ignored. Subsequent calculations of the pressure at the water surface take turbulence into account using eddy viscosity models of varying degrees of complexity. Most of these studies (e.g. Gent & Taylor 1976; Al-Zanaidi & Hui 1984) involve a fairly heavy amount of numerical computatation which, in our opinion, tends to obscure the physics. In particular, field observations (Snyder *et al.* 1981) indicate that for $c < V$ the vertical variation of the perturbation pressure in the air is consistent with potential theory, and it is unclear from the numerical turbulence calculations why the velocity perturbation should be irrotational. The numerical calculations also provide little information on the asymptotic structure of the perturbed flow.

These issues are considered in papers on turbulent flow over water waves by Knight (1977) and on steady turbulent flow over a small hump by Jackson & Hunt (1975). Knight and Jackson & Hunt present linearized asymptotic theories valid for small values of the ratio ϵ of the friction velocity to the characteristic wind speed, and find that the flow outside the viscous sublayer has a two-layer structure consisting of an inner layer of dimensionless thickness ϵ and an outer region in which the velocity perturbation is irrotational to lowest order in ϵ. Despite considerable disagreement between the solutions found by Knight and Jackson & Hunt, their theories agree in predicting a pressure force on the lower boundary in order of magnitude agreement with the numerical results of Gent & Taylor and Al-Zanaidi & Hui and with observations.

Both Knight and Jackson & Hunt use an eddy viscosity assumption as part of their turbulence models. An eddy viscosity approach was also used in a numerical study of turbulent flow over a rigid wavy surface by McLean (1983), and good agreement between measured and computed wall stresses was obtained for flow over small-amplitude waves.

Sykes's (1980) treatment of the problem considered by Jackson & Hunt points out a significant difference between second-order turbulence modelling and the eddy viscosity models used in the paper cited above. If (ρV^2) is used as a pressure scale, eddy-viscosity models predict a dimensionless pressure force of magnitude $O(\epsilon\delta^2)$ on a surface protrusion with slope δ, whereas the second-order closure method used in Sykes's paper yields a pressure force of magnitude $O(\epsilon^2\delta^2)$. A study of the wave-generation problem or of the problem treated by McLean using the model employed by Sykes would therefore predict pressure forces smaller by a factor of ϵ than those calculated using eddy viscosity models or observed in field and laboratory experiments. For this reason we disagree with Sykes's view that the pressure force predicted by eddy-viscosity models is spuriously large, and feel instead that such models are preferable to the current generation of second-order closure models in treating problems of this type.

In the present paper we will present an asymptotic theory for the turbulent flow over water waves based on the use of a turbulence model in which the eddy viscosity is assumed to vary linearly with distance from the water surface. Our reasons for initiating another treatment of this extensively studied problem are to resolve disagreements between the treatments of Knight and Jackson & Hunt by making an independent calculation of the perturbation velocity and pressure and to compare computed wave growth rates with the results of recent field and laboratory observations. We will also be concerned with determining the extent to which the results depend on the turbulence model and with the physical mechanism involved in the wave-generation process.

The analysis given below is based on a scaling argument which shows that the flow structure depends on ϵ, the ratio of the friction velocity to the wind speed, and on the parameter

$$w = \frac{V-c}{V}. \tag{1.4}$$

We assume that ϵ is small, and throughout most of the paper we restrict our attention to the case $|w| \gg O(\epsilon)$, which includes the most rapidly growing waves. For $|w|$ in this range we find that the asymptotic structure consists of an outer flow with lengthscale $L = 1/\alpha$ and a defect layer with lengthscale ϵL, in agreement with other analytical calculations of perturbed turbulent flow. Explicit solutions are given for the perturbation velocity and pressure in each region, and it is shown that the outer solution and the perturbation pressure at the water surface are independent of the eddy-viscosity model up to and including $O(\epsilon)$ terms. The solution for the surface pressure provides our main result, the formula,

$$\beta_{\mathrm{i}} = 2\epsilon\kappa w, \tag{1.5}$$

for β_{i}, the imaginary part of the pressure coefficient in (1.1), in which $\kappa = 0.41$ is Karman's constant.

According to the present theory, the phase lag between the surface pressure and the wave elevation occurs because of a term in the x-component of the turbulent force. Stewart (1974) presents another explanation for the phase shift based on a physical argument about the behaviour of turbulent eddies in flow adjacent to a wavy surface. His expression for the imaginary part of the pressure coefficient agrees precisely with (1.5) if our factor 2κ in that equation is replaced by a factor 1.2.

After allowing for differences in notation, we find that our outer solution and our expression for the pressure coefficient coincide with Knight's, but that our inner solution includes a boundary-layer correction to the outer flow which is present in Jackson & Hunt's solution but which is omitted in Knight's. We have been unable to account for the absence of this part of the solution in Knight's theory because of the lack of detail in his paper. A comparison of our solution with Jackson & Hunt's is even more difficult because, as noted by Sykes, Jackson & Hunt's analysis is not strictly rational in the sense of singular perturbation theory.

Growth rates calculated using (1.5) are in good quantitative agreement with the numerical turbulence-model calculations of Gent & Taylor and Al-Zanaidi & Hui. The present theory is also in satisfactory agreement with observed growth rates for the most rapidly growing waves (Plant 1982; Mitsuyasu & Honda 1982), and yields the intuitively pleasing result that waves travelling against the wind, or more rapidly than the wind, transfer energy and momentum from the water to the air. However, as in other turbulence calculations carried out using eddy-viscosity models, values of the imaginary part of the pressure coefficient calculated here are small compared to the observations of Snyder *et al.* It is unclear at present whether this indicates a deficiency in the theory or in the observations.

2. Derivation of the inner and outer equations

Let T denote time and (X, Y) a set of rectangular coordinates, with Y pointing in the vertical direction. We consider turbulent flow over a wave surface at $Y = L\delta f(X, T)$, where L is the reciprocal of a characteristic wavenumber and δ is a characteristic wave slope. In the present calculation angle brackets denote an ensemble average, and the velocity is expressed as the sum of averaged and random parts $\boldsymbol{u}$ and $\boldsymbol{u}'$. Then, employing dyadic notation (Bird *et al.* 1977, appendix A) and letting π denote the ensemble averaged pressure, we introduce the eddy viscosity ν and a pressure-like variable p through

$$< \boldsymbol{u}'\boldsymbol{u}' > = \tfrac{1}{3} < \boldsymbol{u}' \cdot \boldsymbol{u}' > \boldsymbol{I} - \nu \boldsymbol{E}, \tag{2.1}$$

and

$$\pi = p - \rho g Y - \tfrac{1}{3}\rho < \boldsymbol{u}' \cdot \boldsymbol{u}' > . \tag{2.2}$$

where $\boldsymbol{I}$ is the unit tensor and

$$\boldsymbol{E} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}, \tag{2.3}$$

and we substitute into the averaged continuity and momentum equations to obtain

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.4}$$

and

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}T} + \frac{1}{\rho}\nabla p = \nabla \cdot (\nu \boldsymbol{E}), \tag{2.5}$$

in which $\mathrm{D}/\mathrm{D}T$ denotes the material derivative.

The boundary conditions at the surface are given by

$$\hat{\boldsymbol{s}} \cdot \boldsymbol{u} \to \frac{u_\tau}{\kappa} \log \frac{n}{Y_0}, \quad \frac{D}{DT}(Y - L\,\delta f) = 0, \quad \nu \to \kappa u_\tau n, \tag{2.6}$$

as $Y \to L\,\delta f$, where $\hat{\boldsymbol{s}}$ is a unit vector parallel to the surface, u_τ is the friction velocity, κ is the Kármán constant, Y_0 is the surface roughness length, n is distance measured normal to the water surface, and δu_{s} is the tangential component of the orbital velocity associated with the waves. Here the wind-induced surface drift is neglected, and $u_{\mathrm{s}} = \alpha c f$ for a monochromatic wave with wavenumber α and speed c.

If $\delta = 0$ the governing equations admit a parallel flow solution $p = v = 0$,

$$u = U(Y) = \frac{U_\tau}{\kappa} \log \frac{Y}{Y_0}, \quad \nu = N(Y) = \kappa U_\tau Y, \quad u_\tau = U_\tau, \tag{2.7}$$

where U_τ is a constant. The solution (2.7) can also be expressed in the form

$$U = V\left[1 + \frac{\epsilon}{\kappa} \log \frac{Y}{L}\right], \quad N = \epsilon \kappa V Y, \quad U_\tau = \epsilon V, \tag{2.8}$$

where V is the wind speed at height L above the water and

$$\epsilon = \frac{\kappa}{\log (L/Y_0)} \tag{2.9}$$

is the square root of a drag coefficient in a quadratic drag law relating the tangential surface stress to the wind speed V. Observations indicate that the drag coefficient ϵ^2 has a magnitude of about 10^{-3}, and so ϵ can be regarded as a small parameter.

It is convenient to introduce dimensionless variables by using L as a lengthscale, V as a scale for the rectangular velocity components (u, v) and for the surface drift u_{s}, (L/V) as a timescale, (ρV^2) as a pressure scale, (ϵV) as a scale for the friction velocity, and $(\epsilon L V)$ as a scale for the eddy viscosity. When expressed in dimensionless terms, the definition of $\boldsymbol{E}$, the continuity equation, and the boundary condition on ν take the same form as before, the momentum equation becomes

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}T} + \nabla p = \epsilon \nabla \cdot (\nu \boldsymbol{E}), \tag{2.10}$$

and the boundary conditions on the velocity are given by

$$\hat{\boldsymbol{s}} \cdot \boldsymbol{u} \to u_\tau\left[1 + \frac{\epsilon}{\kappa}\log(n)\right] + \delta u_{\mathrm{s}}, \quad \left(\frac{\partial}{\partial T} + \boldsymbol{u} \cdot \nabla\right)(Y - \delta f) = 0 \quad \text{as } Y \to \delta f. \tag{2.11}$$

Although we agree with Townsend's (1972) view that the critical layer is unimportant in turbulent flows of this type, it should be noted that expressing the dimensionless basic solution in the form

$$U = 1 + \frac{\epsilon}{k}\log(Y), \quad N = \kappa Y, \quad U_\tau = 1, \tag{2.12}$$

does not preclude the existence of a critical layer at which $U = c$ for travelling wave perturbations with dimensionless phase speed c. Instead, (2.12) implies that the critical layer lies outside the viscous sublayer only if c is close to unity, as occurs in Miles's calculation of the phase speed for the most rapidly amplifying waves in his inviscid theory.

Small perturbations to the basic flow are treated most efficiently by using Joseph's theory of domain perturbations (Joseph 1973). As explained by Lebovitz (1982), this consists of introducing a transformation

$$\boldsymbol{X} = \boldsymbol{x} + \delta \boldsymbol{r}(\boldsymbol{x}, t) + O(\delta^2), \quad T = t, \tag{2.13}$$

which maps the boundary $Y = \delta f$ into a plane, and then expanding the dependent variables in a perturbation series of the form

$$F(\boldsymbol{x}, t) = F^{[0]} + \delta[F^{[1]} + \boldsymbol{r} \cdot \nabla_x F^{[0]}] + O(\delta^2), \tag{2.14a}$$

in which all terms are expressed as functions of the new variables $(\boldsymbol{x}, t)$. It follows from the theory set forth by Joseph and Lebovitz that the partial differential equations satisfied by $F^{[k]}$ have the same form in $\boldsymbol{x}$-space as the equations satisfied in $\boldsymbol{X}$-space by the quantities $F^{(k)}$ in the more conventional expansion

$$F(\boldsymbol{x}, T) = F^{(0)}(\boldsymbol{x}, T) + \delta F^{(1)}(\boldsymbol{X}, T) + \ldots, \tag{2.14b}$$

and therefore the boundary can be mapped into a plane without encountering complications involving the metric tensor in the governing partial differential equations.

To carry out the procedure in the present problem, we map the boundary $Y = \delta f$ into $y = 0$ by introducing orthogonal curvilinear coordinates (x, y) through

$$X = x - \frac{\delta}{H} y \frac{\partial f}{\partial x}, \quad Y = \frac{y}{H} + \delta f, \quad T = t, \tag{2.15}$$

where

$$H = (1 + \delta^2 (\partial f / \partial x)^2)^{\frac{1}{2}}, \tag{2.16}$$

and where y is distance measured along a perpendicular to the water surface, and we expand the dependent variables in the form

$$u = U(y) + \delta\left[u^{[1]} + f\frac{\mathrm{d}U}{\mathrm{d}y}\right], \quad \nu = N(y) + \delta\left[\nu^{[1]} + f\frac{\mathrm{d}N}{\mathrm{d}y}\right], \quad u_\tau = 1 + \delta u_\tau^{[1]}, \quad F = \delta F^{[1]}, \tag{2.17}$$

where F denotes either v or p. Here U and N are given by (2.12) with Y replaced by y, and so the terms

$$\left[u^{[1]} + \frac{\epsilon f}{\kappa y}\right], \quad [\nu^{[1]} + \kappa f],$$

denote the $O(\delta)$ contributions to the horizontal velocity and eddy viscosity.

Substituting (2.17) into the governing equations, neglecting $O(\delta^2)$ terms, and omitting the superscripts yields the linearized equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.18}$$

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\mathrm{d}U}{\mathrm{d}y} + \frac{\partial p}{\partial x} = \epsilon\left[N\nabla^2 u + \frac{\mathrm{d}N}{\mathrm{d}y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\mathrm{d}U}{\mathrm{d}y}\right)\right], \tag{2.19}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \epsilon\left[N\nabla^2 v + 2\frac{\mathrm{d}N}{\mathrm{d}y}\frac{\partial v}{\partial y} + \frac{\partial}{\partial x}\left(\nu\frac{\mathrm{d}U}{\mathrm{d}y}\right)\right], \tag{2.20}$$

together with the boundary conditions

$$u+\frac{\epsilon f}{\kappa y}\to u_s+u_\tau U,\quad v\to\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x},\quad \nu+\kappa f\to\kappa u_\tau y, \tag{2.21}$$

as $y\to 0$. We also impose the boundary conditions $(u,v,p)\to 0$ as $y\to\infty$.

In the treatment of a monochromatic wave with dimensional wavenumber α, we let the lengthscale $L = 1/\alpha$ and express f and the dependent variables F as the real parts of

$$f = \exp[\mathrm{i}(x-ct)],\quad F = \hat{F}(y)\exp[\mathrm{i}(x-ct)], \tag{2.22}$$

where the dimensionless phase speed c is the ratio of the dimensional phase speed to V. Substituting (2.22) into the momentum equations shows that the magnitudes of the time-derivative terms are proportional to

$$w = 1-c, \tag{2.23}$$

with an $O(\epsilon)$ error, and that the magnitudes of the turbulent stress terms are proportional to ϵ. Consequently, the ratio of the stress terms to the time-derivative terms is $O(\epsilon/w)$, and the solution has a boundary-layer structure if $|w| \gg O(\epsilon)$. We restrict our attention to this case because the observations cited earlier imply that the most rapidly growing waves have dimensionless phase speeds small compared to unity. Employing standard methods in boundary-layer theory shows that the dimensionless boundary-layer thickness is $O(\epsilon)$.

In treating the outer region we note that the quantity ν in the turbulent stress terms is multiplied by the $O(\epsilon^2)$ quantity $(\epsilon\,\mathrm{d}U/\mathrm{d}y)$, and can be neglected since we intend carrying out the calculation up to and including $O(\epsilon)$ terms. Eliminating p by cross-differentiation and noting also that the flow in the outer region is irrotational to lowest order in ϵ, we find that the perturbation vorticity equation in the outer region becomes

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\nabla^2\Psi+\frac{\epsilon}{\kappa y^2}\Psi = 0, \tag{2.24}$$

correct to $O(\epsilon)$, where Ψ is a stream function defined by

$$u = -\frac{\partial \Psi}{\partial y},\quad v = \frac{\partial \Psi}{\partial x}, \tag{2.25}$$

and that the perturbation pressure in this region solves

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x} = \epsilon\kappa\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-v\frac{\mathrm{d}U}{\mathrm{d}y}, \tag{2.26a}$$

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+\frac{\partial p}{\partial y} = 2\epsilon\kappa\frac{\partial v}{\partial y}, \tag{2.26b}$$

again correct to $O(\epsilon)$.

In our treatment of the inner layer, we express the term involving ν on the right-hand side of the x-momentum equation in the form

$$\epsilon\frac{\partial}{\partial y}\left(\nu\frac{dU}{\mathrm{d}y}\right) = f\left(\frac{\epsilon}{y}\right)^2+\epsilon^2\frac{\partial}{\partial y}\left(\frac{\nu+\kappa f}{\kappa y}\right), \tag{2.27}$$

and we follow Jackson & Hunt by assuming that the perturbation eddy viscosity $(\nu+\kappa f)$ varies linearly with y throughout the inner layer. Then the second term on the right-hand side in (2.27) vanishes, and introducing a stretched inner coordinate $\eta = y/\epsilon$ and defining quantities Q_x and Q_y through

$$u = -\frac{f}{\kappa\eta}+Q_x, \quad v = \left(\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}\right)+\epsilon Q_y, \tag{2.28}$$

in which U is expressed as

$$U = 1+\frac{\epsilon \log(\epsilon)}{\kappa}+\frac{\epsilon \log(\eta)}{\kappa}, \tag{2.29}$$

we obtain the governing equations for the inner layer in the form

$$\frac{\partial}{\partial x}Q_x+\frac{\partial}{\partial \eta}Q_y = 0, \tag{2.30}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)Q_x+\frac{\epsilon}{\kappa\eta}Q_y+\frac{\partial p}{\partial x} = \frac{\partial}{\partial \eta}\left(\kappa\eta\frac{\partial}{\partial \eta}Q_x\right)+\epsilon\kappa\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\right), \tag{2.31}$$

$$\epsilon\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2 f+\frac{\partial p}{\partial \eta} = 0, \tag{2.32}$$

together with the boundary conditions

$$Q_x \to u_\tau U+u_s, \quad Q_y \to 0, \tag{2.33}$$

as $\eta \to 0$.

Although these equations will be solved in the next section, it is worth showing here that the wave growth rate can be determined very simply if we assume that the inner solution for p is the inner expansion of the outer solution. If this is true, the rate at which energy is transmitted to the water is calculated by noting that the $O(\delta)$ contribution to the normal component of velocity at the water surface is given to lowest order in ϵ by $(v-\partial f/\partial x)$, where v is evaluated at $y = 0$, so that the dimensional rate of pressure working W becomes

$$W = -\rho V^3\delta^2\overline{p(v-\partial f/\partial x)}, \tag{2.34}$$

in which p is the outer pressure evaluated at $y = 0$ and the overbar denotes the average over a wavelength. The outer flow is irrotational to lowest order in ϵ and satisfies

$$v = \frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}, \tag{2.35}$$

at $y = 0$, which implies that

$$u = w\cos(x-ct)\,\mathrm{e}^{-y}, \quad v = -w\sin(x-ct)\,\mathrm{e}^{-y}, \tag{2.36}$$

if

$$f = \cos(x-ct). \tag{2.37}$$

It follows that

$$v-\frac{\partial f}{\partial x} = -\left(\frac{1-w}{w}\right)v \tag{2.38}$$

at $y = 0$, which in turn implies that

$$W = \rho V^3\delta^2\left(\frac{1-w}{w}\right)\overline{pv}. \tag{2.39}$$

To evaluate the average on the right-hand side of (2.39), we denote the components of the force on the right-hand side of (2.26) by F_x and F_y, and find that the rate of work done by this force is given by

$$D = uF_x+vF_y = \epsilon\kappa u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-uv\frac{\mathrm{d}U}{\mathrm{d}y}+\epsilon\kappa\frac{\partial}{\partial y}(u^2+v^2). \tag{2.40}$$

The first term on the right-hand side of (2.40) vanishes to lowest order in ϵ and the average of the second term over a wavelength vanishes. Consequently, multiplying (2.26*a*) and (2.26*b*) by u and v, respectively, averaging over a wavelength, and integrating over y yields

$$\overline{pv} = -\int_0^\infty \overline{D}\,\mathrm{d}y = \epsilon\kappa w^2, \tag{2.41}$$

and hence

$$W = \rho V^3 \epsilon\kappa\delta^2 w(1-w). \tag{2.42}$$

Returning temporarily to the use of dimensional variables, we express W in terms of the dimensional phase velocity c to obtain

$$W = \rho V^3 \epsilon\kappa\delta^2 \left(\frac{c}{V}\right)\left(\frac{V-c}{V}\right), \tag{2.43}$$

and note that the growth factor αC_i in (1.1) is given in terms of the energy density E_w of the waves by

$$2\alpha C_\mathrm{i} = \frac{W}{E_w}, \tag{2.44}$$

where α is the dimensional wavenumber and

$$E_\mathrm{w} = \rho_\mathrm{w}\,\delta^2 \frac{c^2}{2\alpha}. \tag{2.45}$$

Therefore the imaginary part of the complex phase velocity is given by

$$C_\mathrm{i} = \epsilon\kappa s c\left(\frac{V-c}{V}\right)\left(\frac{V}{c}\right)^2, \tag{2.46}$$

subject to verification of our assumption that the inner solution for p is the inner expansion of the outer solution. As can be seen, (2.41) is the key equation in the derivation because it shows that the phase shift between the wave-induced pressure perturbation and the surface elevation is induced by a term in the gradient of the Reynolds stress.

3. Solution of the inner and outer equations

We now restrict our attention to travelling wave perturbations to the basic flow, and we employ dimensionless variables and the notation of (2.22). For a monochromatic progressive wave the surface drift term u_s in (2.33) is replaced by the dimensionless wave speed c and the operator $(\partial/\partial t + \partial/\partial x)$ by iw, where w is defined by (2.23).

Turning first to the outer problem, we find that the solution $\hat{\Psi}$ of (2.24) satisfying $\hat{\Psi} \to 0$ as $y \to \infty$ is a multiple A of $W_{0,m}(2y)$, where $W_{0,m}$ is Whittaker's confluent hypergeometric function (Whittaker & Watson 1965, pp. 339–340) and where

$$m = \tfrac{1}{2}\left(1 - \frac{4\epsilon}{kw}\right)^{\frac{1}{2}}. \tag{3.1}$$

Substituting (3.1) into the integral representation of the Whittaker function given on page 340 of Whittaker & Watson and expanding the result in powers of ϵ yields

$$W_{0,m}(2y) = \mathrm{e}^{-y} + \frac{\epsilon}{\kappa w}\mathrm{e}^{y}\mathrm{Ei}(-2y) + O(\epsilon^2), \tag{3.2}$$

where $-\mathrm{Ei}(-y)$ is the exponential integral (Lebedev 1972, pp. 30–32). The form of the $O(1)$ solution of the inner equations obtained below implies that the $O(1)$ contribution to the inner expansion of $\hat{\Psi}$ must equal w and that the constant term in the $O(\epsilon)$ contribution vanishes. Use of the small-argument expansion of (3.2) then implies that the multiplying constant A is given by

$$A = w-\frac{\epsilon}{\kappa}(\gamma+\log(2)), \tag{3.3}$$

correct to $O(\epsilon)$, where γ is Euler's constant.

The inner expansion of the outer solution can now be determined by multiplying (3.2) by (3.3) and by expressing the solutions for the perturbation velocity components and pressure in terms of the inner variable η. Carrying out this calculation yields

$$\hat{u} = w-\frac{1}{\kappa\eta}-\frac{\epsilon\log(\epsilon)}{\kappa}-e\left[w\eta+\frac{\log(\eta)+2\gamma+2\log(2)-1}{\kappa}\right], \tag{3.4}$$

$$\hat{v} = \mathrm{i}\left[w+\frac{\epsilon\log(\epsilon)}{\kappa}+\epsilon\frac{\log(\eta)}{\kappa}-\epsilon w\eta\right], \tag{3.5}$$

and

$$\hat{p} = -w^2+\epsilon w\left[\frac{2\gamma+2\log(2)}{\kappa}+2\mathrm{i}\kappa+w\eta\right]. \tag{3.6}$$

The solution of the inner equations is found by expanding the dependent variables in the series

$$F = F^{(0)}+\epsilon\log(\epsilon)F^{(1)}+\epsilon F^{(2)}+\dots, \tag{3.7}$$

which yields

$$\hat{Q}_x^{(0)} = w, \quad \hat{Q}_y^{(0)} = -\mathrm{i}w\eta, \quad \hat{p}^{(0)} = -w^2, \quad \hat{u}_\tau^{(0)} = 1-2\mathrm{c}, \tag{3.8}$$

$$\hat{Q}_x^{(1)} = -\frac{1}{\kappa}, \quad \hat{p}^{(1)} = 0, \quad \hat{u}_\tau^{(1)} = -\frac{2}{\kappa}w. \tag{3.9}$$

Carrying the solution to $O(\epsilon)$ shows that

$$\hat{Q}_x^{(2)} = c_1-w\eta-\frac{1}{\kappa}\log(\eta)+R(\eta), \tag{3.10a}$$

$$\hat{p}^{(2)} = w\left(\frac{1}{\kappa}-c_1+2\mathrm{i}\kappa+w\eta\right), \tag{3.10b}$$

where

$$c_1 = -\frac{2\gamma+2\log(2)-1}{\kappa}, \tag{3.11}$$

and where R solves

$$\frac{d}{\mathrm{d}\eta}\left(\kappa\eta\frac{\mathrm{d}R}{\mathrm{d}\eta}\right) = \mathrm{i}wR, \tag{3.12}$$

subject to

$$R\to\hat{u}_\tau^{(2)}-c_1+\frac{2w}{\kappa}\log(\eta) \quad (\eta\to 0), \quad R\to 0 \quad (\eta\to\infty). \tag{3.13}$$

Equations (3.12) and (3.13) are solved by

$$R = -\frac{2w}{\kappa}K_0(b\eta^{\frac{1}{2}}), \tag{3.14}$$

where K_0 is a modified Bessel function,

$$b = 2\left(\frac{|w|}{\kappa}\right)^{\frac{1}{2}} \exp\left[\left(\tfrac{1}{4}\mathrm{i}\pi\right)\operatorname{sgn}(w)\right], \tag{3.15}$$

and

$$\hat{u}_T^{(2)} = \frac{1+2\gamma(1-2c)+2w\log(|w|/\kappa)-2\log(2)+\mathrm{i}\pi|w|}{\kappa}, \tag{3.16}$$

and this completes the solution for the inner region correct to $O(\epsilon)$. As noted previously, our outer solution agrees with Knight's, while his inner solution omits a boundary-layer term of the type given here by $R(\eta)$.

It can be seen by inspection that the inner solution for the perturbation pressure is the inner expansion of the outer solution. This verifies the assumption made at the end of the previous section, and so (2.46) provides the solution for the imaginary part of the complex phase velocity in (1.1). The pressure coefficient β in that equation is equal to the value of $\hat{p}$ at the water surface, and is obtained by setting $\eta = 0$ in (3.6).

Before going on to compare our solution for the wave growth rate with the results of other theories and with observations, we need to consider how our results would be modified if we had employed a more sophisticated eddy-viscosity model. The most straightforward way of resolving this issue is to scale the perturbation eddy viscosity using the assumptions common to standard one-equation and two-equation eddy-viscosity models and to repeat the calculation carried out above. The results of this lengthy procedure show that the outer solution and the surface pressure are independent of the turbulence model up to and including $O(\epsilon)$ terms, and that the $O(1)$ and $O(\epsilon\log(\epsilon))$ inner solutions are also independent of the model. However, the $O(\epsilon)$ x-momentum equation in the inner region contains a term on the right-hand side involving the perturbation eddy viscosity which is omitted in the present theory, and therefore the boundary-layer correction calculated above involving the modified Bessel function would take a different form if we had used a more complicated turbulence model. The pressure coefficient, however, is independent of the turbulence model to the order of approximation considered here.

4. Comparison with other studies

In order to compare our results with observations and with other calculations, we revert to the use of dimensional variables and find from the above analysis that the pressure coefficient β defined in (1.1) is given by

$$\beta = -\left(\frac{V-c}{V}\right)^2 + 2\epsilon\frac{V-c}{V}\left(\frac{\gamma+\log(2)}{\kappa}+\mathrm{i}\kappa\right). \tag{4.1}$$

Taking the imaginary part of (4.1) yields the formula (1.5) for β_{i}, and substituting (1.5) into (1.3) yields the imaginary part of the complex phase velocity given by (2.46), the expression for which shows that waves travelling against the wind or at speeds greater than the wind transfer energy from the water to the air. For amplifying waves, we substitute the expression for C_{i} into the definition,

$$\zeta = 2\frac{C_{\mathrm{i}}}{c}, \tag{4.2}$$

of Miles's growth factor to obtain the important result

$$\zeta = 2\epsilon\kappa s\frac{V-c}{V}\left(\frac{V}{c}\right)^2. \tag{4.3}$$

An analysis of the meteorological data given in the paper by Snyder *et al.* cited earlier suggests that the roughness length can be calculated using a Charnock relation of the form

$$Y_0 = \frac{(U_\tau)^2}{ga}, \tag{4.4}$$

with $a = 352.3$. This value for a is much larger than results obtained for a fully developed sea (cf. Stewart 1974, figure 1), but appears to be appropriate for the initial phases of wave growth. Use of (2.9), (4.4), and the relation between the speed of deep-water waves and the lengthscale $L = 1/\alpha$ shows that the parameter,

$$\lambda = \frac{2\kappa s}{\epsilon}, \tag{4.5}$$

can be expressed in the form

$$\lambda = 2s \log\left[a\left(\frac{c}{U_\tau}\right)^2\right], \tag{4.6}$$

and for waves such that $c \ll V$, (4.3) can be approximated by

$$\zeta = \lambda\left(\frac{U_\tau}{c}\right)^2, \tag{4.7}$$

with λ given by (4.6). The papers by Mitsuyasu & Honda and Plant cited in § 1 express empirical equations for wave growth rates in this form.

Observations suggest that the most rapidly growing waves have phase speeds of the order of $10U_\tau$, and in order to compare our results with measured growth rates and with other theoretical calculations, we assume $c = 10U_\tau$ in (4.6) and obtain the estimate $\lambda = 0.026$ if we let the density ratio s take the value 1.25×10^{-3}. Al-Zanaidi & Hui find that for the roughness lengths appropriate for field data, their numerical calculations of the growth rate are consistent with (4.7) provided that

$$\lambda = \lambda_{\mathrm{AH}} = 0.031, \tag{4.8}$$

This is in reasonably good agreement with the present theory. Gent & Taylor's calculations for the case of fixed roughness length provide a value

$$\lambda_{\mathrm{GT}} = 0.023, \tag{4.9}$$

for small-amplitude waves with $\epsilon = 0.05$. Our theory implies that $\lambda = 0.021$ for this value of ϵ, and again the agreement is good. The observational papers provide the values

$$\lambda_{\mathrm{P}} = 0.04 \pm 0.02, \quad \lambda_{\mathrm{MH}} = 0.054, \tag{4.10}$$

where λ_{P} is taken from Plant's survey of field and laboratory observations and λ_{MH} from Mitsuyasu & Honda's wave-tank experiments. Our value $\lambda = 0.026$ agrees reasonably well with Plant's correlation, but appears to underestimate the growth rate for wave-tank experiments.

Unfortunately, the degree of agreement between (4.3) and the measurements of Snyder *et al.* for larger values of c is very poor. According to Snyder *et al.*, the growth factor ζ can be expressed in terms of the wind speed U_5 at elevation 5 m in the form

$$\zeta = (0.2 \text{ to } 0.3)\left(\frac{U_5 - c}{c}\right) \quad (c < U_5 < 3c), \tag{4.11}$$

for waves travelling in the wind direction. Using the estimate of the Charnock constant given earlier and the average wind speed measured by Snyder *et al.*, we find if $c = \frac{1}{2}U_5$ our estimate for ζ is only about 40% as large as the value given by (4.11), and that the lack of agreement between our theory and observations is greater for larger values of c. The numerical calculations of Gent & Taylor and of Al-Zanaidi & Hui are in even poorer agreement with (4.11) for values of c greater than $\frac{1}{2}U_5$, and therefore either the measurements are inaccurate or the present type of turbulence modelling fails for some reason when applied to values of the phase speed comparable in magnitude to the wind speed. In the paper by McLean cited in §1, the agreement between his theory and observations was much better for flow over small-amplitude waves on a rigid surface than for flow over large-amplitude waves. This suggests that curvature effects may cause errors in eddy viscosity models in dealing with the comparatively large-amplitude waves usually observed when the phase speed is close to the wind speed, but further investigation is needed to verify this conjecture.

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